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LETTER TO THE EDITOR

Geometric origin of central charges

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Abstract. The complete set of $N(N-1)$ central charge generators for the $D=4$ N-extended super-Poincaré algebra is obtained by suitable contraction of the OSp($2N$; 4) superalgebra. The superspace realisations of the spinorial generators with central charges are derived. The conjugate set of $N(N-1)$ additional bosonic superspace coordinates is introduced in a unique and geometric way.

The central charges were introduced originally by Haag *et al* (1975) as a mathematical possibility‡, but recently they appeared to be an important enlargement of the fundamental super-Poincaré algebra. Already the study of $N=2$ extended supersymmetry shows (see e.g. Sohnius *et al* 1980) that in some cases only the presence of central charges allows for the existence of non-trivial (i.e. describing interacting theory) field representations. It is also believed that the difficulties with the formulation of $N \geqslant 3$ extended supergravities are related to the lack of proper understanding of the role of degrees of freedom generated by central charges.

The aim of this Letter is to show that the central charge generators have a clear meaning as the contracted generators from the internal symmetry sector. If we assume that the linearly realised group on extended super-Poincaré generators is SL(2, C)$\times$ U(N), the supergroup which provides such a contraction can be uniquely determined as OSp($2N$; 4). In our contraction scheme *all* the bosonic generators survive the contraction limit: the translation generators are obtained from the four non-compact generators describing Sp(4)/SL(2, C) and $N(N-1)$ central charges are obtained from the compact generators in the coset O($2N$)/U(N). From such a construction it follows that the central charges transform under SU(N) as the antisymmetric second-rank tensor $Z_{ab} = X_{ab} + Y_{ab}$, denoted by (2, 0).

We would like to mention here that central charges are scalars under the 'exact' internal symmetry of the S matrix, but can transform under spontaneously broken internal symmetry. One should assume therefore that the U(N) symmetry of N-extended supersymmetry is spontaneously broken in the presence of central charges. The reduced 'exact' symmetry group for different choices of central charge generators has been recently calculated by Ferrara *et al* (1980).

Let us recall that the superalgebra OSp($2N$; 4) has the bosonic sector $g = g_1 \oplus g_2$ where

$$g_1 = \mathrm{Sp}(4, R) \simeq \mathrm{O}(3,2) \qquad \text{(space–time symmetries)},$$

$$g_2 = \mathrm{O}(2N) \qquad \text{(internal symmetries)}$$

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and the fermionic sector is generated by $2N$ four-component Majorana charges†. We introduce five 4×4 real matrices, describing the Majorana representation of the O(3, 2) Clifford algebra

$$\{\Gamma_A, \Gamma_B\} = 2g_{AB}, \qquad g_{AB} = \mathrm{diag}(-1, 1, 1, 1, -1).$$

The supercharges Q^i_α, extending the Lie algebra $\mathrm{Sp}(4, R)\oplus\mathrm{O}(2N)$ to the superalgebra $\mathrm{OSp}(2N; 4)$, satisfy the following relation:

$$\{Q^i_\alpha, Q^j_\beta\} = \delta^{ij}(\Gamma^\mu\Gamma_0)_{\alpha\beta}P_\mu + m[\delta^{ij}(\sigma^{\mu\nu}\Gamma_0\Gamma_5)_{\alpha\beta}M_{\mu\nu} + g(\Gamma_0\Gamma_5)_{\alpha\beta}T^{ij}] \tag{1}$$

where $i, j = 1\ldots 2N$, $\alpha, \beta = 1\ldots 4$, $\mu, \nu = 0, 1, 2, 3$, $\sigma_{\mu\nu} = \frac{1}{4}[\Gamma_\mu, \Gamma_\nu]$ and $T^{ij} = -T^{ji} = \tau^{ij}_r T_r$ describe $\mathrm{O}(2N)$ generators expanded in the $2N\times 2N$ matrix basis, denoted by τ^{ij}_r ($r = 1\ldots N(2N-1)$; $\tau^{ij}_r = -\tau^{ji}_r$). For dimensional reasons we have introduced the mass parameter m and dimensionless coupling g characterising respectively the de Sitter radius and the Yang–Mills coupling in the $\mathrm{O}(2N)$ sector. The covariance relations are as follows (cf Zumino 1977):

$$(1/\mathrm{i})[P_\mu, Q^i_\alpha] = \tfrac{1}{2}m(\Gamma_\mu\Gamma_5)\alpha^\beta Q^i_\beta, \tag{2a}$$

$$(1/\mathrm{i})[M_{\mu\nu}, Q^i_\alpha] = (\sigma_{\mu\nu})\alpha^\beta Q^i_\beta, \tag{2b}$$

$$(1/\mathrm{i})[T_r, Q^i_\alpha] = g\tau^{ik}_r Q^k_\alpha. \tag{2c}$$

The generator of 'curved translations' P_μ satisfies the Z_2-graded de Sitter algebra

$$(1/\mathrm{i})[P_\mu, P_\nu] = -m^2 M_{\mu\nu}. \tag{3}$$

Let us introduce the Z_2 grading in the internal symmetry sector by the following split of the $\mathrm{O}(2N)$ Lie algebra

$$T_{ij}\ \mathrm{O}(2N):\qquad \underbrace{\left(\begin{array}{c|c} A & S \\ \hline -S & A \end{array}\right)}_{\mathrm{U}(N)} \oplus \underbrace{\left(\begin{array}{c|c} X & Y \\ \hline Y & -X \end{array}\right)}_{\frac{\mathrm{O}(2N)}{\mathrm{U}(N)}} \tag{4}$$

where $N\times N$ real matrices satisfy the relations

$$A = -A^{\mathrm{T}}, \qquad X = -X^{\mathrm{T}},$$
$$S = S^{\mathrm{T}}, \qquad Y = -Y^{\mathrm{T}}. \tag{5}$$

Further, we introduce the generalised Weyl–Majorana spinors

$$Q^\pm = \tfrac{1}{2}(1 \pm \Gamma_5\Omega)Q, \qquad \Omega^{ij} = -\Omega^{ji}, \qquad \Omega^2 = -1, \tag{6}$$

satisfying the condition $Q^\pm = \pm\Gamma_5\Omega Q^\pm$. Choosing

$$\Omega = \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix},$$

one can write

$$Q^\pm = \frac{1}{2}\begin{pmatrix} Q^a \pm \Gamma_5 Q^{a+N} \\ \mp\Gamma_5(Q^a \pm \Gamma_5 Q^{a+N}) \end{pmatrix}. \tag{7}$$

† An alternative equivalent formulation uses $2N$ two-component complex Weyl spinors (see e.g. Ferrara 1977).

The unconstrained N-component Majorana spinor describing $Q^{\pm}$ is denoted $Q_\alpha^{a\pm}$ ($a = 1 \ldots N$):

$$Q_\alpha^{a\pm} = \tfrac{1}{2}(Q_\alpha^a \pm \Gamma_5 Q^{a+N}). \tag{8}$$

The commutation relations (1) split as follows ($m' = gm$):

$$\{Q_\alpha^{a+}, Q_\beta^{b+}\} = \delta^{ab}(\Gamma^\mu \Gamma_0)_{\alpha\beta} P_\mu + m'[(\Gamma_0)_{\alpha\beta} X^{ab} + (\Gamma_0 \Gamma_5)_{\alpha\beta} Y^{ab}], \tag{9a}$$

$$\{Q_\alpha^{a+}, Q_\beta^{b-}\} = m\delta^{ab}(\Sigma^{\mu\nu}\Gamma_0\Gamma_5)_{\alpha\beta} M_{\mu\nu} + m'[(\Gamma_0)_{\alpha\beta} A^{ab} + (\Gamma_0\Gamma_5)_{\alpha\beta} S^{ab}], \tag{9b}$$

$$\{Q_\alpha^{a-}, Q_\beta^{b-}\} = \delta^{ab}(\Gamma^\mu \Gamma_0)_{\alpha\beta} P_\mu + m'[(\Gamma_0)_{\alpha\beta} X^{ab} - (\Gamma_0 \Gamma_5)_{\alpha\beta} Y^{ab}]. \tag{9c}$$

The relations ($2a, b$) are valid also for $Q_\alpha^{a\pm}$, but the relation ($2c$) only remains valid for the generators belonging to the U(N) subgroup of O($2N$). Let us observe that

$$\begin{aligned} &[\tau_r, \Omega] = 0 \qquad \text{iff } \tau_r \in \mathrm{U}(N), \\ &\{\tau_r, \Omega\} = 0 \qquad \text{iff } \tau_r \in \mathrm{O}(2N)/\mathrm{U}(N). \end{aligned} \tag{10}$$

Writing $X^{ab} + \Gamma_5 Y^{ab} = \tau_K^{ab}(X_K + \Gamma_5 Y_K)$ ($K = 1 \ldots \frac{1}{2}N(N-1)$) one obtains

$$(1/i)[X_K, Q_\alpha^{a\pm}] = g\tau_K^{ab} Q_\alpha^{b\mp}, \qquad (1/i)[Y_K, Q_\alpha^{a\pm}] = \pm g\tau_K^{ab}(\Gamma_5 Q^{b\mp})_\alpha. \tag{11}$$

It is easy to check that the whole superalgebra OSp($2N$; 4) can be decomposed into the following four sectors,

$$\begin{array}{llll} L_0 & L_1 & L_2 & L_3 \\ M_{\mu\nu} & Q_\alpha^{a+} & P_\mu & Q_\alpha^{a-} \\ A^{ab}, S^{ab} & & X^{ab}, Y^{ab} & \end{array} \tag{12}$$

and satisfies the Z_4-grading relations†

$$\langle L_i, L_j \rangle \subset L_{i+j} \qquad (i, j, i+j \bmod 4) \tag{13}$$

where L_0, L_2 are bosonic and L_1, L_3 fermionic sectors. The grading (13) implies that one can perform the rescaling

$$L_0 \to L_0, \qquad L_1 \to \frac{1}{\sqrt{R}} L_1, \qquad L_2 \to \frac{1}{R} L_2, \qquad L_3 \to \frac{1}{\sqrt{R}} L_3, \tag{14}$$

and consider the contraction limit $R \to \infty$ in consistency with the Jacobi identities. The rescaled generators satisfy the relations ($9a, c$), but we obtain $\{Q_\alpha^{a+}, Q_\beta^{b-}\} \sim 1/R$ and also $[X_K, Q_\alpha^{a\pm}] \sim 1/R$, $[Y_K, Q_\alpha^{a\pm}] \sim 1/R$. The O($2N$) superalgebra is split by the grading (13) into the Z_2-graded symmetric Riemannian pair (U(N), O($2N$)/U(N)) with the generators from O($2N$)/U(N) transforming under U(N) as the antisymmetric 2-tensor (2, 0). One can also write the algebra of de Sitter type (see (3)) describing

† The Z_4 gradings were introduced by Lukierski (1980) for the description of the supersymmetric generalisation of a Riemannian symmetric pair ($H, G/H$): the so-called super-Riemannian quadruple. The Z_4 gradings of physically interesting supergroups were discussed by the authors in Lukierski and Rytel (1980), where also the grading (12) is given. Already in Lukierski and Rytel (1980) the relation of internal symmetry generators from the sector L_2 to central charges in the contraction limit $R \to \infty$ was pointed out.

the internal symmetry sector. One obtains

$$[X_K, X_L] \sim \frac{1}{R^2} \mathrm{U}(N), \qquad [X_K, X_L] \sim \frac{1}{R^2} \mathrm{U}(N), \qquad [Y_K, Y_L] \sim \frac{1}{R^2} \mathrm{U}(N), \tag{15}$$

where U(N) denotes the element of the U(N) algebra.

It is easy to see therefore that in the contraction limit $R \to \infty$ the $N(N-1)$ generators X_K, Y_K become Abelian ones, and in the extended super-Poincaré algebra they appear as the central charges. Because if $R \to \infty$ the sectors Q_α^{a+} and Q_α^{a-} decouple, one can restrict the fermionic sector to L_1 (or L_3). If we denote $Q_\alpha^a \equiv Q_\alpha^{a+}$, one obtains in the limit $R \to \infty$ from the sectors (L_0, L_1, L_2) the $D = 4$ extended super-Poincaré algebra with $N(N-1)$ real central charges.

Before contraction in the OSp($2N$; 4) superalgebra the generators X_k, Y_k describe non-Abelian or curved 'precentral' charges; in order to obtain the conventional description of central charges one has to perform two steps.

(a) *The contraction limit.* The generators become Abelian, but still do not commute with the internal symmetry sector U(N) (they can be called Abelian 'precentral' charges).

(b) *Reduction of internal symmetry group U(N) to its subgroup commuting with* Z^{ij}. Such a reduction for different choices of the representations with non-trivially represented X_{ab}, Y_{ab} was discussed recently in Ferrara *et al* (1980); see also Lopuszanski and Wolf (1982).

It is already known (see e.g. Sohnius 1978) that the superspace realisation of supersymmetries with central charges requires additional bosonic coordinates of superspace. In particular, one can obtain central charges from the translations in extra bosonic dimensions (Olive 1979, Osborn 1979), but in such a case one introduces the 'big' Lorentz group $O(D-1, 1)$ $(D > 3)$ and the breaking mechanism into physical space–time and the internal sector is not justified. In the derivation of central charges presented here the additional bosonic variables acquire a clear geometric meaning: they describe complex 'curved' translations on the coset O($2N$)/U(N), and in the limit $R \to \infty$ one obtains $N(N-1)$ additional bosonic variables u_{ab}, v_{ab} ($u_{ab} = -u_{ba}$, $v_{ab} = -v_{ba}$), with the translations generated by the central charge generators.

The N-extended superspace variables before the contraction parametrise the coset

$$K = \mathrm{OSp}(2N; 4)/\mathrm{SL}(2; C) \times \mathrm{U}(N). \tag{16}$$

Using the exponential parametrisation, one can write ($\bar{\theta} = \theta^{\mathrm{T}}\Gamma_0$)

$$K = \exp(X_\mu P^\mu + Q_\alpha^{a+}\bar{\theta}_\alpha^{a+} + Q_\alpha^{a-}\bar{\theta}_\alpha^{a-}) \exp(X_{ab}u^{ab} + Y_{ab}v^{ab}). \tag{17}$$

Applying well known techniques of nonlinear realisations on coset spaces (see e.g. Zumino 1977, Gursey and Marchildon 1978), one obtains the transformation formulae of the superspace coordinates (X_μ, $\theta_+^{\alpha a}$, $\theta_-^{\alpha a}$, u^{ab}, v^{ab}) e.g. under the supertranslation $\exp(Q_\alpha^{a+}\,\varepsilon_+^{\alpha a})$. Using the rescaled superalgebra, one can calculate the tranformations up to order $1/R$. Describing the change of superfield coordinates as the result of the action of the superspace generator $\varepsilon^+ Q^+$, one obtains

$$Q_\alpha^{a+} = \frac{\partial}{\partial\theta_\alpha^a} - \tfrac{1}{2}(\bar{\theta}^a\Gamma_\mu)_\alpha \frac{\partial}{\partial x_\mu} + m'\left(\bar{\theta}_\alpha^b \frac{\partial}{\partial u_{ba}} + (\bar{\theta}^b\Gamma_5)_\alpha \frac{\partial}{\partial v_{ba}}\right) + O\left(\frac{1}{R}\right) \tag{18}$$

where we have put $\theta_\alpha^a \equiv \theta_\alpha^{a+}$. Calculating the anticommutator of the generators (19), one obtains

$$\{Q^a_\alpha, Q^b_\beta\} = \delta^{ab}(\Gamma_\mu\Gamma_0)\frac{\partial}{\partial x_\mu} + 2m'\left((\Gamma_0)_{\alpha\beta}\frac{\partial}{\partial u_{ab}} + (\Gamma_0\Gamma_5)_{\alpha\beta}\frac{\partial}{\partial v_{ab}}\right) + O\left(\frac{1}{R}\right). \qquad (19)$$

The formula (20) in the limit $R \to \infty$ coincides with the extension of the Salam–Strathdee superspace realisation of the N-extended super-Poincaré algebra with $N(N-1)$ central charges†.

We wuld like to add the following remarks.

(a) We learned when our work was finished that Howe and Lindstrom (1981) and Kalosh (1981) introduced an additional 56 bosonic coordinates for $N = 8$ supergravity in order to incorporate into the curved superspace formalism the E_7 'hidden symmetry' group. In our framework these additional variables are determined uniquely, and one can show that they carry the linear realisations of the earlier discovered hidden symmetry E_7 (see e.g. Julia 1980). At present we are not able to determine which 'hidden symmetries' occur at given N using purely geometric arguments; however, a look at the additional variables and linear realisations of hidden symmetries at any given N leads to the conclusion that they are 'well fitted' to each other.

(b) The additional bosonic coordinates can be used for the definition of mass operator which depends on the Casimirs of internal and hidden symmetry groups. These coordinates transform only under internal symmetries, and represent a new version of the 'old' isotopic space concept.

(c) We discuss in this note only rigid extended supersymmetries. Our way of introducing central charges if applied to supergravity requires local gauging of OSp(2N; 4) symmetry. The methods developed by McDowell (see e.g. McDowell 1980) and by the supergroup approach (see e.g. d'Auria *et al* 1980) should be useful in deriving the supergravity actions with central charges.

(d) Because the number of new bosonic coordinates increases quadratically with N, it is reasonable to look for *composite superspaces*, by supersymmetric extension of the Penrose framework with 'elementary' twistors and 'composite' space–time coordinates. The outline of such a generalisation, with more details provided for $N = 1$, has been presented recently by one of the authors (Lukierski 1981).

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References

d'Auria R, Fré P and Regge T 1980 *Riv. Nuovo Cimento* vol 3 12
Ferrara S 1977 *Phys. Lett.* **69**B 481
Ferrara S, Savoy C A and Zumino B 1980 *CERN preprint TH*3010
Gursey F and Marchildon L 1978 *Phys. Rev.* D **17** 2038
Haag R, Loupuszanski J and Sohnius M 1975 *Nucl. Phys.* **B88** 257
Howe P and Lindstrom U 1981 *Nucl. Phys.* B **181** 487

† The formula (19) in the limit $R \to \infty$ was also presented by J G Taylor during the supergravity workshop at ICTP, Trieste (4–6 May 1981). For the necessity of central charges in extended supergravity theories see also Rivelles and Taylor (1981).

Julia B 1980 *Superspace and Supergravity—Proc. Nuffield Workshop* (Cambridge: CUP) p 331
Kalosh R 1981 *Talk at Supergravity Workshop* 4–6 *May ICTP Trieste*
Lopuszanski J and Wolf M 1982 in preparation
Lukierski J 1980 *Proc. IXth Colloquium on Group Theoretic Methods, Cocoyoc Lecture Notes in Physics* vol 135 p 580
—— 1981 *Lecture at Winter School in Karpacz, February* 1981 *Wroclaw University preprint no* 534 *April* 1981
Lukierski J and Rytel L 1980 *University of Leuven preprint* KUL-TF-80/11 unpublished
McDowell S W 1980 *Superspace and Supergravity—Proc. Nuffield Workshop* (Cambridge: CUP) p 257
Olive D 1979 *Nucl. Phys.* B **153** 1
Osborn M 1979 *University of Cambridge preprint* DAMP 79/4
Rivelles V and Taylor J G 1981 *Phys. Lett.* **104** B 131
Sohnius M 1978 *Nucl. Phys.* B **138** 109
Sohnius M, Stelle K S and West P C 1980 *Proc. Europhysics Conference—Unification of fundamental interactions—Erice March* 1980 (New York: Plenum)
Zumino V B 1977 *Nucl. Phys. B* **127** 189